of the foil. In some experiments the diameter of foil was equal 15 mm and the possibility of flame propagation along the side of the specimen was excluded. In this case a round hole in the foil was observed after the combustion wave running through the metal foil. The hole diameter was equal to 11 mm. It was assumed that the foil was destroyed after the combustion wave passed through the foil. Special experiments were carried out with an ebonite or plexiglas specimen instead of lower powder specimen. In these cases an extinction took place and the foil was unharmed.

A special experiment was also made to investigate the effect of cohesion of the foil with the side coating of the specimens (series V). In this experiment the foil area was greater than the cross section of the specimen. The edge of the foil was squeezed by two special plexiglas plates which were fitted on propellant specimens and were fixed on both specimens. The clamp is illustrated schematically in Fig. 2. No essential influence of the clamp upon the value of critical pressure interval was observed. It is necessary to note that if the combustion front does not pass through a metal foil, a fine slab of the unburned powder remains at the metal. (At $p = 20$ atm the foil thickness is less than 0.03 mm.) It means that the extinction takes place before the combustion front reaches the upper surface of the metal foil.

When the combustion wave runs through a metal foil a delay of ignition is observed. The lower specimen ignition does not occur simultaneously with arrival of the luminous combustion front at the foil. If the pressure increases the ignition delay time decreases.

The experimental results in the case of series VII are shown in Fig. 3. In this series of experiments both powder specimens had no side coating. In Fig. 3 the function $\mathcal{P}(p)$ has a specific form.

It is necessary to note that when a foil thickness is given, "the probability of combustion wave running through a foil" increases with pressure and the combustion rate also increases, as is known. Therefore the characteristic thermal layer in a specimen decreases.

It is possible to give a qualitative explanation of the phenomenon of combustion wave running through a foil. The thermal diffusivity of foil is higher than that of the powder and we can assume the foil temperature does not depend on the spatial coordinate. In this approximation the balance of thermal energy of the foil is as follows:

$$\lambda(\zeta_+ - \zeta_-) = \rho c l(\partial T/\partial t)$$

where ρ is foil density, c is foil specific heat, λ is thermal conductivity of powder, l is foil thickness, T is the mean foil temperature, ζ_+, ζ_- are gradients of temperature at the upper and lower surfaces of the foil in the specimens.

There is a jump of the temperature gradient at the foil. Therefore at the moment when the combustion front reaches the upper foil surface the temperature gradient at the lower foil surface is smaller than at the upper one. According to the theory, for example,[6] the propagation of combustion wave in solid propellant (powder N) is possible only if the temperature gradient at the burning surface exceeds a certain critical minimum value ζ_m. If the temperature gradient at the lower foil surface is smaller than the critical value the ignition of the lower specimen is impossible. If the temperature gradient at the lower foil surface is greater than the critical value the combustion wave penetrates through the foil. Since the range of allowable temperature gradients at the burning surface increases with increasing pressure, the ζ_- value remains less than ζ_m at $p < p_*$ and the ignition of the lower specimen is thus impossible. If $p > p_*$ then $\zeta_- > \zeta_m(p)$ and the combustion wave penetrates through the metal foil.

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Subcritical and Supercritical Boundary Layers

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Momentum integral techniques have been used for many years to solve the laminar and turbulent boundary-layer equations and have met with a considerable measure of success. In attempting to apply such methods to an ever-increasing range of Mach numbers and wall temperature ratios, some difficulty has arisen because under certain supersonic or hypersonic cold wall conditions the boundary layer tends to thin in an adverse pressure gradient and it is difficult to match this behavior to the development of the compression process in the external stream. Crocco[1] was the first to recognize this dilemma and he classified boundary layers as either subcritical or supercritical, depending on whether the displacement thickness increased or decreased in an adverse (i.e., positive) pressure gradient. This classification was adopted in the Crocco-Lees[2] paper and more recently in the Lees-Reeves[3] method of solving the laminar boundary-layer equations. Lees and Reeves found it necessary to introduce a jump from supercritical to subcritical behavior in order to reach separation with hypersonic cold wall flows.

A great deal of discussion has centered on the physical causes of these difficulties and much ingenuity has been used in pushing the supercritical boundary just out of sight. For example, it has been shown[2] that the choice of boundary-layer thickness (δ) has a considerable influence on the position of the critical line; the larger the value of δ the more hypersonic stream tubes are included and the more supercritical the boundary layer becomes. The neglect of the normal pressure gradient (certainly an important feature of the real flow) also has been cited as the main cause of difficulty, and Holden[4] has shown how the introduction of centrifugal effects effectively removes the supercritical boundary from regions of interest. The main aim of this Note is to re-examine a few of the various momentum integral methods and demonstrate that the critical boundary depends entirely on the formulation

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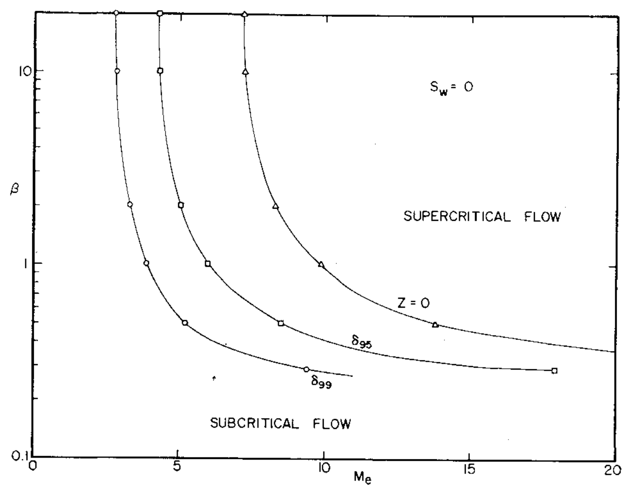

Fig. 1 Critical boundary ($D = 0$) as a function of the boundary-layer thickness criterion.

of the method to be employed, i.e., it is a mathematical difficulty devoid of physical significance and distinct from Crocco's original definition.

One of the most ambitious uses of the momentum integral technique has been made by Lees and Reeves.[3] They employ the integral equations of momentum and moment of momentum together with a coupling equation linking the boundary-layer growth to the development of the external pressure field. The energy equation is not used, the velocity and temperature profiles being linked as in the Cohen-Reshotko catalogue[5] of similarity solutions. The set of equations they solve conveniently may be written in matrix form as

$$\begin{array}{ccccl} \dfrac{d\delta_t^*}{dX} & \dfrac{\delta_t^*}{Me}\cdot\dfrac{dMe}{dX} & \delta_t^*\dfrac{dH}{dX} & & \dfrac{\gamma_\infty}{a_\infty Me\delta_t^*} \\ H & 2H+1 & 1 & = & P \quad (1) \\ J & 3J+2SwT^* & dJ/dH & = & R \quad (2) \\ B & f & 1 & = & -h \quad (3) \end{array}$$

The notation is that of Ref. 3. The functions P, R and h are

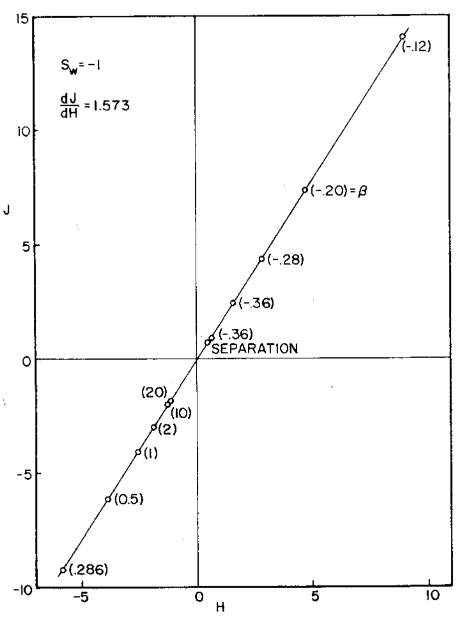

Fig. 2 J vs H for similar boundary layers ($S_w = -1$).

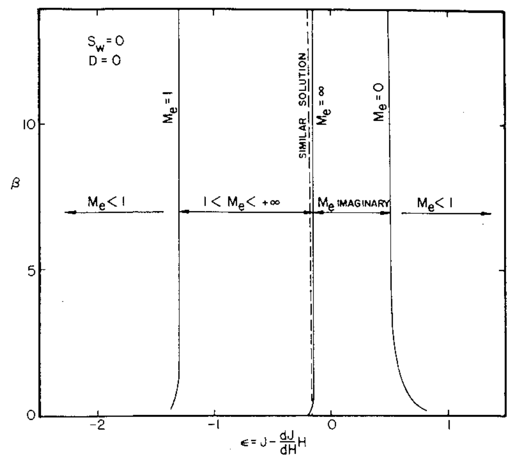

Fig. 3 Existence of the critical boundary as a function of ϵ.

related to the three unknowns δ_t^*, Me, and H through the parameters a using Ref. 5. The solution of these three nonlinear ordinary differential equations can be written as

$$(\delta_t^*/Me)\cdot(dMe/dX) = (1/R_{e_{\delta_t^*}}) \times [N_1(\delta_t^*,M_e,H)/D(H,M_e)] \quad (4)$$

$$\delta_t^*(dH/dX) = (1/R_{e_{\delta_t^*}})\cdot(N_2/D) \text{ etc.} \quad (5)$$

The critical boundary is defined by $D = 0$ where D is evaluated from the 3×3 determinant shown in Eqs. (1–3), i.e.,

$$D = [(1 + me)/m_e][(3J + 2SwT^*) - (2H + 1) \times (dJ/dH)] + \epsilon[(2H + 1) - f] \quad (6)$$

where

$$\epsilon = J - H(dJ/dH)$$

The boundary, $D = 0$, may be plotted in the m_e vs 'a' plane or, since $m_e = [(\gamma - 1)/2]M_e^2$ and 'a' is related to β (similarity profile parameter), in the M_e vs β plane as shown in Fig. 1 for the adiabatic wall condition $S_w = 0$. The choice of boundary-layer thickness (δ_{95} or δ_{99}) enters through the parameter δ. The curve $Z = 0$ in Fig. 1 corresponds to $\delta = \delta^*$ so that the entrainment term in the coupling equation is zero. It can be seen that the choice of δ has a considerable influence on the boundary position.

Now, as Hankey and Cross[6] pointed out J/H is nearly constant and, in fact, $J = kH$ is a particularly good approximation, especially for cold wall conditions, as Fig. 2 shows ($\epsilon \approx 0$). Making this approximation reduces Eq. (6) to

$$m_e/(1 + m_e)\cdot D = J(1 - H^{-1} + 2SwT^*/J) \quad (7)$$

Hence, the boundary $D = 0$, given by

$$1 - H^{-1} + 2SwT^*/J = 0 \quad (8)$$

is independent of both Mach number and choice of boundary-layer thickness. Substitution of values for H, T^*, and J calculated by Christian[7] ($-1 < Sw < +1$) demonstrate that D never reaches zero.

For adiabatic conditions in which a critical boundary was observed (Fig. 1), ϵ has a value of about -0.17. Since ϵ (the value of the J intercept on the $J - H$ curve, Fig. 2) of zero produced no critical boundary, Eq. 6 was solved for various ϵ values for $D = 0$. The results (Fig. 3) show that no critical boundary exists for $\epsilon > -0.15$. Since the assumption that J is only a function of H is an approximation, it would seem unlikely that any physical significance can be placed on the boundary for which $D = 0$ due to the ease at which the boundary can be relocated.

The additional approximation $J = kH$ in an already approximate method completely removes the critical boundary from regions of practical interest. However, this change also turns the Lees-Reeves method into a local similarity technique in the sense that dMe/dX becomes tied to β and is no longer a function of Me. This can be demonstrated by rewriting Eqs. (1–3) as follows:

$$\begin{array}{cccccc} \dfrac{d\delta_t^*}{dX} & \dfrac{\delta_t^*}{Me}\dfrac{dMe}{dX} & \delta_t^* & \dfrac{dH}{dX} & \dfrac{\gamma_\infty}{a_\infty Me\delta_t^*} & \\ H & 2H+1 & 1 & & = P & (9) \\ H & 3H+2SwT^*/k & 1 & & = R/k & (10) \\ B & f & 1 & & = -h & (11) \end{array}$$

or subtracting Eq. (9) from Eq. (10)

$$\begin{array}{ccccc} H & 2H+1 & 1 & P & (9) \\ 0 & H+(2SwT^*/k)-1 & 0 & -P & (10a) \\ B & f & 1 & -h & (11) \end{array}$$

or

$$\begin{array}{ccccc} H & 2H+1 & 1 & P & (9) \\ 0 & 1 & 0 & \dfrac{RH-PJ}{JH\left(1-H^{-1}+\dfrac{2SwT^*}{J}\right)} & (10b) \\ B & f & 1 & -h & (11) \end{array}$$

Equation (10b) states that $(\delta_t^*/Me)\cdot(dMe/dX) = fn(\beta)$ for a given Sw, and evaluating the final determinant gives the boundary D as

$$D = H - B \equiv -(1+me)/me \qquad (12)$$

so that D is never zero, as stated earlier.

The modification $J = kH$ has reduced the moment of momentum equation into an auxiliary equation of the form

$$\frac{\delta_t^*}{Me}\cdot\frac{dMe}{dX} = fn(\beta,Sw) = (RJ^{-1} - PH^{-1}) \times \left(1 - H^{-1} + \frac{2SwT^*}{J}\right)^{-1} \qquad (13)$$

and the method, as shown by Hankey,[6] becomes identical in principle to that of Cohen and Reshotko.[9] Their auxiliary equation was expressed in its simplest form as

$$(\delta_t^*/Me)\cdot(dMe/dX) = -n(p,Sw) \qquad (14)$$

with

$$n = [PH - A/(B - H^{-1} - 2)]1/H^2R_{e_{\delta t^*}} \qquad (15)$$

where A and B are constants. Comparing Eqs. (15) and (10b) suggests that

$$RH = kA \qquad (16)$$

and

$$2SwT^*/J = B - 3 \qquad (17)$$

Lees and Reeves pointed out that a one-parameter method might be inadequate for tackling very cold wall flows and indicated some possible lines of development. Holden[8] extended the Lees-Reeves method by including the energy equation. This allows the velocity and temperature profiles to be disconnected. Holden found a critical boundary for $Sw = -0.8$ that again can be removed by the assumption that $J = kH$.

Conclusions

The method of Lees and Reeves unhooks the velocity profiles from the pressure gradient parameter by employing a moment of momentum equation. This formulation unfortunately introduces a critical boundary at which the gradients of all the flow variables can become infinite. This boundary has no physical significance and its position is extremely sensitive to the numerical value of the parameters used and to the choice of the boundary-layer thickness. Above all, the boundary is the result of the formulation of the problem. A slight modification to the formulation converts the method into one similar in principle to that of Cohen and Reshotko. Although sacrificing the unhooking of velocity profile from pressure gradient, this modification completely removes the critical boundary from the domain of interest.

Similar remarks apply to the method of Holden. The same modification of assuming that the ratio of the boundary-layer kinetic energy thickness to momentum thickness is constant reduces the moment of momentum equation to an auxiliary equation as used by Cohen and Reshotko. The great benefit is the removal of the critical boundary.

In the past the critical boundary has been linked to a physical definition of subcritical and supercritical flow, whereas in fact these are two different things. The critical boundary defined by the characteristic determinant of the equation set going to zero is dependent on the formulation of the problem. It is quite easy to formulate the problem in such a way that no such mathematical difficulty exists. The physical distinction based on $d\delta^*/dp \gtrless 0$ is a real one and may well be important in all techniques for solving the behavior of compressible boundary layers.

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